TECHNICAL NOTE: STEADY FLOW IN A CURVED PIPE WITH A COAXIAL CORE

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SUMMARY

In this paper we study the steady annular flow of a viscous fluid into an annular pipe and discuss the effects of the size of the core on the flow properties.

KEYWORDS: annular flow; curved pipe

Annular flow is an important feature in double-pipe heat exchangers and in chemical mixing and drying machinery. This type of flow involving heat transfer has been studied by Karahalios[1] and Choi and Park,[2] while previous investigation has been mainly focused on the axial flow between concentric cylinders.[3,4]

In the present work we consider an annular pipe coiled in a circle of radius L about the axis Oz. Let a be the radius of the inner pipe and ka ($k > 1$) the radius of the outer pipe. The flow is steady, incompressible and fully developed and the ratio ka/L is assumed small. We use a toroidal co-ordinate system r^*, θ, ϕ (Figure 1) to describe the equations of motion. In this system the velocity components are U, V and W respectively. Let $-\partial p/\partial\phi$ be the pressure gradient. Introducing the non-dimensional variables

$$r^* = kar, \qquad U = \nu u/ka, \qquad V = \nu v/ka, \qquad W = \nu[L/2(ka)^3]^{1/2}w$$

and setting

$$u = \frac{1}{r}\frac{\partial f}{\partial\theta}, \qquad v = -\frac{\partial f}{\partial r}$$

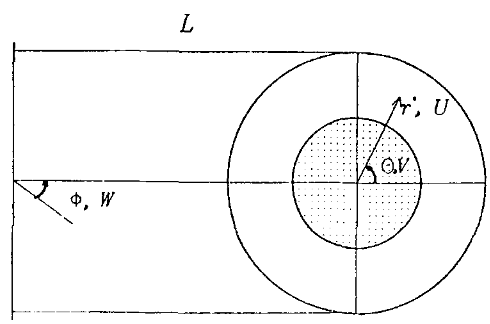

Figure 1. Toroidal co-ordinate system

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so that the equation of continuity is identically satisfied, we obtain the equations of motion

$$\nabla^2 f = -\Omega, \tag{1}$$

$$\nabla^2 w + \frac{1}{r}\frac{\partial(f, w)}{\partial(r, \theta)} = -D, \tag{2}$$

$$\nabla^2 \Omega + \frac{1}{r}\frac{\partial(f, \Omega)}{\partial(r, \theta)} = w\left(\frac{\partial w}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial w}{\partial \theta}\cos\theta\right), \tag{3}$$

where ∇^2 is the Laplacian operator in polar co-ordinates,

$$D = \left(\frac{2ka}{L}\right)^{1/2} \frac{G(ka)^3}{\mu v}$$

is the Dean number of the flow and

$$G = -\frac{1}{L}\frac{\partial p}{\partial \phi}.$$

In the previous notation, f is the streamfunction of the flow and Ω is the vorticity. The conditions satisfied by the parameters of the flow are

$$w = f = \frac{\partial f}{\partial r} = 0 \quad \text{at } r = 1 \text{ and } 1/k.$$

In addition, the symmetry condition about the line $\theta = 0, \pi$ implies that

$$f(r, -\theta) = -f(r, \theta), \qquad w(r, -\theta) = w(r, \theta), \qquad \Omega(r, -\theta) = -\Omega(r, \theta),$$

while $f = \Omega = \partial W/\partial\theta = 0$ on the line itself. The equations of motion have been solved numerically according to a method described by Allen and Southwell[5] and worked out by Dennis,[6,7] as follows. Equation (2) is separated into two equations

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - u\frac{\partial w}{\partial r} = A(r, \theta) - D,$$

$$\frac{\partial^2 w}{\partial \theta^2} - rv\frac{\partial w}{\partial \theta} = -r^2 A(r, \theta),$$

where $A(r, \theta)$ is an unknown function. After lengthy manipulations, $A(r, \theta)$ is eliminated and the following equation for w is derived:

$$m_1 w_1 + m_2 w_2 + m_3 w_3 + m_4 w_4 - m_0 w_0 + h^2 D, \tag{4}$$

where

$$m_1 = 1 + \frac{1}{2r_0} - \frac{1}{2}u_0 h + \frac{u_0^2 h^2}{8}, \qquad m_2 = \lambda^2\left(\frac{1}{r_0^2} - \frac{v_0 g}{2r_0} + \frac{v_0^2 g^2}{8}\right),$$

$$m_3 = 1 - \frac{h}{2r_0} + \frac{1}{2}u_0 h + \frac{u_0^2 h^2}{8}, \qquad m_4 = \lambda^2\left(\frac{1}{r_0^2} + \frac{v_0 g}{2r_0} + \frac{v_0^2 g^2}{8}\right),$$

$$m_0 = 2 + \frac{2\lambda^2}{r_0^2} + \frac{1}{4}h^2(u_0^2 + v_0^2), \qquad \lambda = \frac{h}{g}$$

and the subscripts follow the Southwell notation, in which all quantities at the point (r_0, θ_0) and the neighbouring points $(r_0 + h, \theta)$, $(r_0, \theta_0 + g)$, $(r_0 - h, \theta_0)$ and $(r_0, \theta_0 - g)$ are denoted by the subscripts 0, 1, 2, 3 and 4 respectively. Treating equation (1) in the same way, we obtain

$$m_1\Omega_1 + m_2\Omega_2 + m_3\Omega_3 + m_4\Omega_4 - m_0\Omega_0 - \frac{1}{2}hw_0\left((w_1 - w_3)\sin\theta_0 + \frac{\lambda}{r_0}(w_2 - w_4)\cos\theta_0\right) = 0. \quad (5)$$

The boundary conditions for Ω are

$$(2 + h)\Omega(1, \theta) = \frac{6f(1 - h, \theta)}{h^2} - \Omega(1 - h, \theta)$$

on the outher boundary $r = 1$ and

$$(2 - hk)\Omega_0 = \frac{6}{h^2}f\left(\frac{1}{k} + h, \theta\right) - \Omega\left(\frac{1}{k} + h, \theta\right).$$

for $r = 1/k$. Finally, equation (3), approximated by central differences about the point (r_0, θ_0), takes the form

$$\left(1 + \frac{h}{2r_0}\right)f_1 + \left(1 - \frac{h}{2r_0}\right)f_3 + \frac{\lambda^2}{r_0^2}(f_2 + f_4) - 2\left(1 + \frac{\lambda^2}{r_0^2}\right)f_0 + h^2\Omega_0 = 0, \quad (6)$$

with boundary conditions $f = 0$ on $r = 1$ and $1/k$. Equations (4)–(6) were solved numerically by the SOR method at all internal points of the upper annular region $1/k \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \pi$. The iterative scheme was repeated until the adopted criterion of accuracy for w,

$$\max\left|1 - \frac{w^s(r, \theta)}{w^{s+1}(r, \theta)}\right| \leqslant 5 \times 10^{-5},$$

was satisfied. The superscript s denotes successive iterates. For the other two variable Ω and f the same criterion of accuracy was adopted. The sequence was terminated when all quantities had converged to limits.

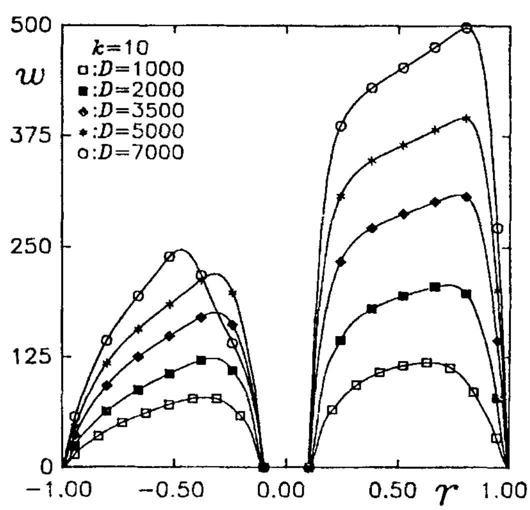

Figure 2. Variation in w with r

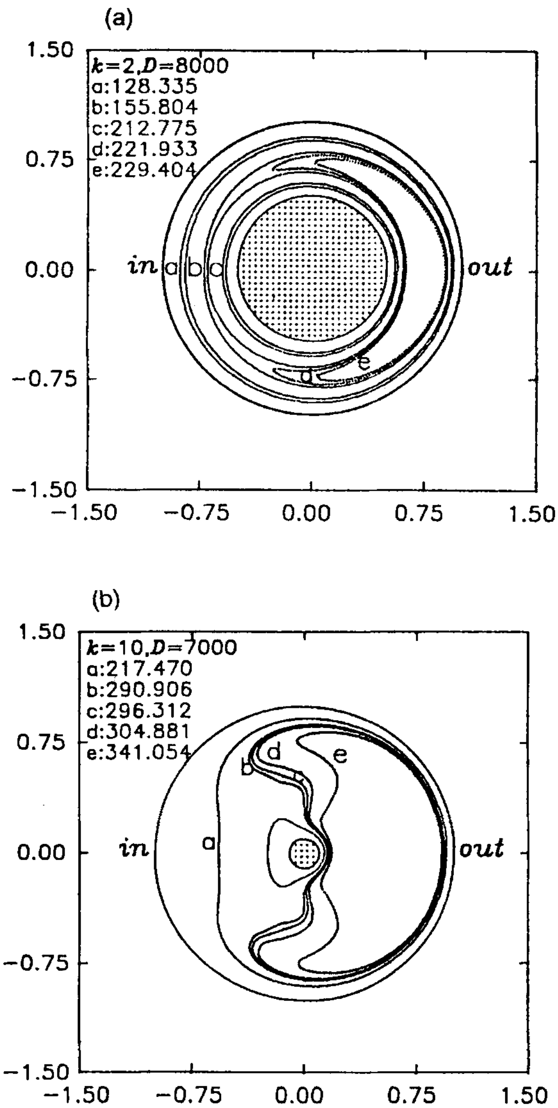

Figure 3. Isovelocity contours for k=(a) 2 and (b) 10

In Figure 2 we show the axial velocity profile along the line of symmetry $\theta = 0, \pi$ for $k = 10$ and for various values of the Dean number D. The form of these curves is in satisfactory agreement with the corresponding curves taken for flow in a plain curved tube. The axial velocity contours are shown in Figure 3(a) for a large core radius and in Figure 3(b) for a smaller core radius. In the first of these plots it is realized that there develops a central inviscid region in which the axial isovelocity lines become parallel over a significant part of this region. For larger axial–core radius the isovelocity curves close to the boundaries are circles concentric to them, while the centrifugal force drives the fluid to the outer part of the bend, thus forming separate loops there.

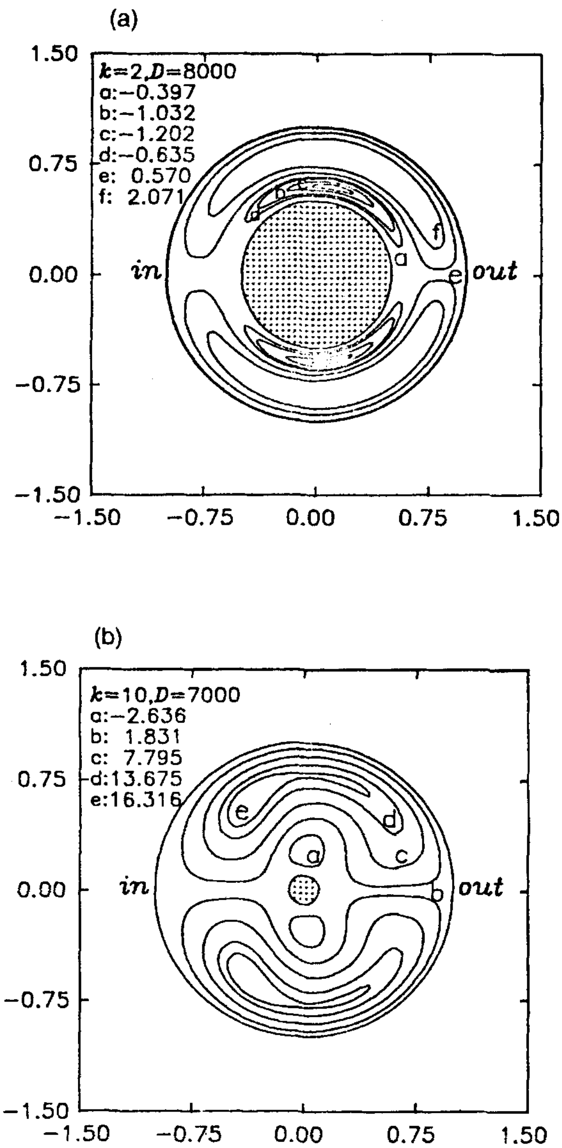

Figure 4. Secondary flow streamline pattern for $k=$ (a) 2 and (b) 10

The curves of constant streamfunction f are shown in Figures 4(a) and 4(b) for $k=2$ and 10 respectively. The formation of the four vortices is a consequence of the viscous character of the Stokes boundary layers that are formed along the two boundaries. In the outer layer the pressure gradient is not balanced by the centrifugal force acting on the fluid. In fact, the pressure gradient remains unchanged while w tends to zero as the pipe wall is approached. As a result, a secondary flow is generated within this layer. The fluid moves from the outside of the bend to the inside along the pipe wall and closes its cycle by moving along the line $\theta = 0, \pi$. Along this line the recirculating fluid of the upper half meets at $\theta = \pi$ the recirculating fluid of the lower half. The two jets impact and move along

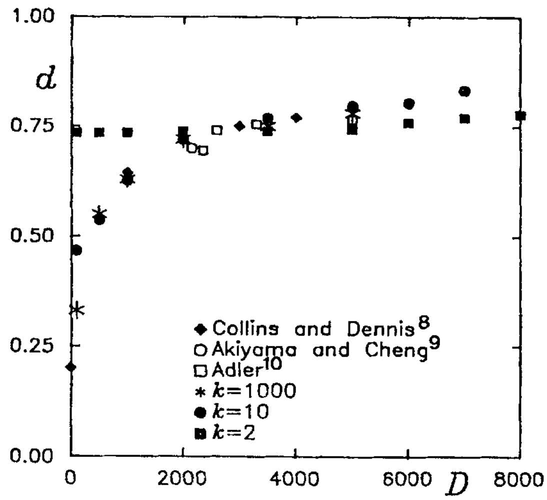

Figure 5. Variation in position d of w_{max} with Dean number D

the pre-referred line of symmetry. The generation of the inner vortex close to the axial core is again attributed to the non-balance of the pre-referred forces. The fluid is moving along the core wall from $\theta = 0$ to π and at an intermediate point it reverses its direction of motion since its strength is overwhelmed by the strength of the outer vortex.

In Figure 5 we show the variation in the position of the maximum axial velocity w_{max} with D for various values of the radius ratio k. The maximum axial velocity occurs on the line of symmetry $\theta = 0, \pi$. In this figure, results given by Collins and Dennis,[8] Akiyama and Cheng[9] and Adler[10] are also presented for comparison. Here d denotes the non-dimensional distance of the position of w_{max} from the centre of the cross-section of the pipe. The agreement of the results of the present study with those of Collins and Dennis[8] is good.

In conclusion, the presence of a core affects the flow properties, especially for a large Dean number. The formation of the pair of inner vortices suggests that instability may occur above some finite Dean number. The question of bifurcation and stability will be addressed in a future study.

APPENDIX: NOTATION

a	radius of core
D	modified Dean number
f	streamfunction
G	constant pressure gradient
h, g	grid sizes
k	radius ratio
L	radius of curvature of pipe
p	pressure
r, θ, ϕ	dimensionless toroidal components
U, V, W	dimensional components of velocity
u, v, w	dimensionless components of velocity

Greek letters and other symbols

λ	h/g
μ	coefficient of viscosity
ν	kinematic viscosity
ρ	density of fluid
Ω	dimensionless vorticity
∇	Laplacian operator

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